

A photonic crystal realization of a phase driven two-level atom

B. M. Rodríguez-Lara,¹ Alejandro Zárate Cárdenas,¹

Francisco Soto-Eguibar,¹ and Héctor Manuel Moya-Cessa¹

¹*Instituto Nacional de Astrofísica, Óptica y Electrónica. Luis Enrique Erro 1,
Santa María Tonantzintla, Puebla, Mexico 72840.**

Abstract

We propose a set of photonic crystals that realize a nonlinear quantum Rabi model equivalent to a two-level system driven by the phase of a quantized electromagnetic field. The crystals are exactly soluble in the weak-coupling regime and their dispersion relation is discrete. The system is diagonalized by normal modes equivalent to a dressed state basis. In the strong-coupling regime, we use perturbation theory and find that the dispersion relation is continuous and give the normal modes of the crystal in terms of continued fractions. We show that these photonic crystals allow state reconstruction in the form of coherent oscillations in the weak-coupling regime.

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I. INTRODUCTION

Photonic crystals as classical simulators of quantum processes have been the focus of attention in recent years.^{1–11} In particular, it has been shown that the so-called quantum Rabi model describing the interaction of a two-level system with a quantum field may be realized by photonic superlattices.¹² The quantum Rabi model in the weak-coupling regime, i.e. the Jaynes-Cummings model,¹³ describes a variety of quantum mechanical systems that have been experimentally implemented; e.g. cavity-quantum electrodynamics (cavity-QED)¹⁴, ion traps¹⁵ and circuit-QED.¹⁶ Strong-coupling is not feasible in a majority of simple quantum optical systems but photonic crystals provide a classical realization of the quantum model in all coupling regimes.¹²

In quantum optics, diverse non-linear models describing the interaction between a two-level system and a quantum field have been proposed as deformations of the Jaynes-Cummings model.^{17,18} One example of these nonlinear models is the Buck-Sukumar (BS) model where the atom-field coupling depends on the intensity of the quantum field.¹⁹ The BS model, which is exactly soluble and does not have a feasible experimental representation, unless it is classically realized in a couple of binary photonic crystals where the coupling depends linearly on the position of the waveguide, helps in understanding the apparition of collapses and revivals of the two-level inversion in the radiation-matter systems.

In the following, we propose a semi-infinite photonic crystal realization of a non-linear model describing a system where the atom-field coupling depends only on the phase of the quantum field; something that is missing in the literature up to our knowledge. Then, we find the exact dispersion curves and normal modes of the waveguide lattice in the weak-coupling regime. In the strong-coupling regime, the dispersion relation is continuous and we find the normal modes as continued fractions. The transition from discrete to continuous spectrum, appearing in our photonic crystal, does not show in the spectra of Rabi^{20–22} and BS¹⁹ models which are always discrete. Thus, parameter sets delivering coherent oscillations in Rabi or BS models only produce coherent oscillations in the weak-coupling regime of our model.

II. THE MODEL AND ITS PHOTONIC CRYSTAL ANALOGUE

Let us consider the Hamiltonian describing a two-level system driven by just the phase of a quantum field,

$$\hat{H} = \omega_f \hat{a}^\dagger \hat{a} + \frac{\omega_0}{2} \hat{\sigma}_z + \lambda \left(e^{i\hat{\phi}} + e^{-i\hat{\phi}} \right) \hat{\sigma}_x, \quad (1)$$

where the exponential of the quantum phase operator is given by the Susskind-Glogower operator²³

$$e^{i\hat{\phi}} \equiv \hat{V} = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}} \hat{a}. \quad (2)$$

Where the field mode of frequency ω_f is described by the annihilation(creation) operators $\hat{a}(\hat{a}^\dagger)$, the two-level system of transition frequency ω_0 by Pauli matrices $\sigma_{x,y,z}$, and their interaction by the real coupling λ . It is possible to separate this system in two uncoupled Hamiltonians,

$$\hat{H}_\pm = \omega_f \hat{n} \mp \frac{\omega_0}{2} (-1)^{\hat{n}} + \lambda \left(\hat{B} + \hat{B}^\dagger \right), \quad (3)$$

belonging to one of two parity chain basis,

$$|+, n\rangle = \hat{B}^{\dagger n} |0, g\rangle, \quad (4)$$

$$|-, n\rangle = \hat{B}^{\dagger n} |0, e\rangle, \quad (5)$$

defined such that parity,

$$\hat{\Pi} = -\sigma_z (-1)^{\hat{n}}, \quad (6)$$

is conserved, $\langle \pm, n | \hat{\Pi} | \pm, n \rangle = \pm$; the bases annihilation(creation) operator is given by $\hat{B} = \hat{V} \hat{\sigma}_x (\hat{B}^\dagger = \hat{V}^\dagger \hat{\sigma}_x)$ and the number operator is defined as $\hat{n} | \pm, m \rangle = m | \pm, m \rangle$.

By defining the general state,

$$|\psi_\pm\rangle = \sum_{j=0}^{\infty} \mathcal{E}_j^{(\pm)} | \pm, j \rangle, \quad (7)$$

the equations of motion for any given initial state under the dynamics given by Hamiltonian (1) are reduced to the differential set

$$i\partial_t \mathcal{E}_j^{(\pm)} = \left[\omega_f j \mp \frac{\omega_0}{2} (-1)^j \right] \mathcal{E}_j^{(\pm)} + \lambda \left(\mathcal{E}_{j-1}^{(\pm)} + \mathcal{E}_{j+1}^{(\pm)} \right), \quad (8)$$

where the shorthand notation ∂_t has been used for the partial derivative with respect to t . This differential set is equivalent, up to a phase and substituting $t \rightarrow z$, to that describing the propagation equation of a classical field through a photonic waveguide lattice. In this equivalent photonic waveguide lattice, \mathcal{E}_j is the amplitude of the field at the j th waveguide, the waveguides are homogeneously coupled, and the refraction indices grow proportional to ω_f and to their position on the lattice plus a position depending bias proportional to $\omega_0/2$.

For the sake of simplicity, hereby we will refer to the photonic crystals as H_+ or H_- , depending on the sign of Eq.(8). In order to construct these photonic crystals, one can choose to either implement a static, Fig. 1(a), or dynamic, Fig. 1(b), relation between parameters ω_f and ω_0 ; bending the waveguides along a circle introduces an index gradient inversely proportional to the wavelength of the impinging light.¹²

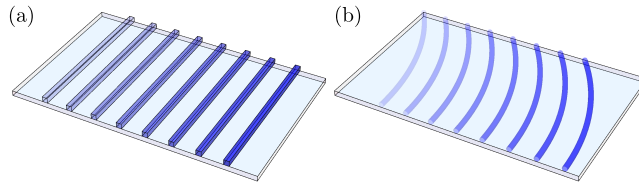


FIG. 1. (Color online) Two schemes to produce the set of two photonic crystals realizing the phase driven two-level atom. A semi-infinite set of homogeneously coupled waveguides where the refraction index behaves as the function $n_j^{(\pm)} \propto \omega_f j \mp \frac{\omega_0}{2} (-1)^j$. (a) Straight waveguides produce a static relation between parameters ω_f and ω_0 . (b) Circularly bent waveguides produce a response to the incident frequency of light producing a dynamic relation between ω_f and ω_0 .

III. DISPERSION RELATION

Up to our knowledge and means, it is not possible to find an exact dispersion relation for the photonic crystals described above but it is possible to separate the relevant coupling parameter in two regimes, weak and strong, in order to obtain some results. In the first of these regimes, we can borrow techniques from quantum optics and find an exact dispersion relation. While on the last, we can only deal with the problem through perturbation theory.

A. Weak-coupling regime: $\lambda \ll \omega, \omega_0$.

In the weak coupling regime one can find the exact spectrum and normal modes of each one of the photonic crystals described by the differential set (8) by taking a step back and implementing the rotating wave approximation in the Hamiltonian of the system,

$$\hat{H}_{RWA} = \omega_f \hat{a}^\dagger \hat{a} + \frac{\omega_0}{2} \hat{\sigma}_z + \lambda \left(e^{i\hat{\phi}} \hat{\sigma}_+ + e^{-i\hat{\phi}} \hat{\sigma}_- \right), \quad (9)$$

before establishing the classical analogue. Then, it is simpler to find the spectrum and normal modes in this representation by using the basis set $\{|n, e\rangle, |n+1, g\rangle\}$ belonging to the manifold with $n+1$ excitations. This leads to the discrete spectrum,

$$E_{\pm, n} = \omega_f \left(n + \frac{1}{2} \right) \pm \frac{\Omega}{2}, \quad \Omega = \sqrt{\delta^2 + 4\lambda^2}, \quad (10)$$

where the proper states are given by

$$|n, \pm\rangle = \alpha_\pm |n, e\rangle + \beta_\pm |n+1, g\rangle, \quad (11)$$

with

$$\frac{\alpha_\pm}{\beta_\pm} = \frac{-\delta \pm \Omega}{2\lambda}. \quad (12)$$

Note that $|0, g\rangle$ is an eigenstate of the Hamiltonian with energy $E_{-,0} = -\omega_0/2$.

In our photonic crystals, Eq. (8), we can approximate the dispersion relation, equivalent to the discrete spectrum found above, by proposing a collective proper mode and realizing that the three-term recurrence relations can be summarized by the tridiagonal matrix,

$$H_{\pm, w} = H_0^{(\pm)} + P, \quad (13)$$

$$\left(H_0^{(\pm)} \right)_{i,j} = \left[\omega_f j \mp \frac{\omega_0}{2} (-1)^j \right] \delta_{i,j}, \quad (14)$$

$$(P)_{i,j} = \lambda (\delta_{i,j+1} + \delta_{i+1,j}), \quad (15)$$

where the notation $(M)_{i,j}$ stands for the (i, j) th term of Matrix M and the symbol $\delta_{a,b}$ is Kronecker's delta. As $\lambda \ll \omega_f, \omega_0$, we can treat matrix P as a perturbation on matrix H_0 and find the eigenvalues of H_\pm up to second order corrections as the first order correction is equal to zero. Thus, we obtain the approximated dispersion relation,

$$\omega(q)^{(\pm)} \approx \omega_f q \mp \frac{\omega_0}{2} (-1)^q \left(1 + \frac{4\lambda^2}{\omega_0^2 - \omega_f^2} \right). \quad (16)$$

Figure 2 shows good agreement between the dispersion relation given by the exact eigenvalues in the rotating wave approximation, Eq. (10), and the perturbation approach, Eq. (16), at zero and second order.

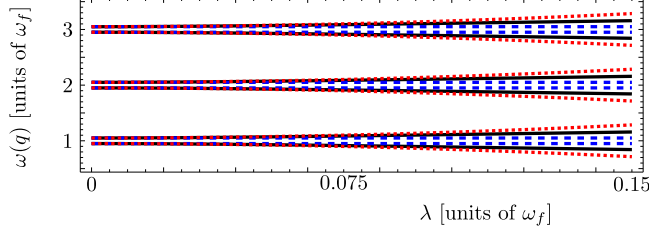


FIG. 2. (Color online) A segment of the dispersion relation in the weak-coupling regime. Exact closed form from rotating wave approximation (solid black), 0th order perturbation (dashed blue) and second order perturbation (dotted red) are shown. We have used a detuning given by $\omega_0 = 1.1 \omega_f$.

B. Strong-coupling regime: $\lambda \gg \omega, \omega_0$.

In the case when the coupling parameter is larger than the field and transition frequencies, sometimes also called deep-coupling regime, it is possible to write the three term recurrence as

$$H_{\pm, S} = H_0 + P_{\pm}, \quad (17)$$

$$(H_0)_{i,j} = \lambda (\delta_{i,j+1} + \delta_{i+1,j}), \quad (18)$$

$$(P_{\pm})_{i,j} = \left[\omega_f j \mp \frac{\omega_0}{2} (-1)^j \right] \delta_{i,j}. \quad (19)$$

Notice that lead order matrix is diagonalized via $H_0 = V \Lambda V^{-1}$, where the diagonal matrix Λ contains the values of the dispersion relation in its diagonal and the matrix V has as columns the coefficients of the normal modes given by

$$(V)_{j,q} = U_j \left(\frac{\mu(q)}{2\lambda} \right), \quad \mu(q) \in \mathbb{R}, \quad (20)$$

where the function $U_n(x)$ is the n th Chebyshev polynomial of the second kind evaluated at x ; i.e., the dispersion relation for this case is continuous.

At this point it is irrelevant, but first order correction for the dispersion relation delivers,

$$\omega(q)^{\pm} \approx \mu(q) + \int_0^{\infty} d\mu(q) \frac{\sum_{k=0}^{\infty} \left[U_k \left(\frac{\mu(q)}{2\lambda} \right) \right]^2 [\omega_f k \mp \frac{\omega_0}{2} (-1)^k]}{\sum_{j=0}^{\infty} \left[U_j \left(\frac{\mu(q)}{2\lambda} \right) \right]^2}. \quad (21)$$

IV. COLLECTIVE MODES

For any given set of parameters, decomposition in normal modes delivers the three term recurrence mentioned before,

$$\left[a_0^{(\pm)} - \omega(q) \right] c_0^{(\pm,q)} + \lambda c_1^{(\pm,q)} = 0, \quad (22)$$

$$\left[a_j^{(\pm)} - \omega(q) \right] c_j^{(\pm,q)} + \lambda (c_{j-1}^{(\pm,q)} + c_{j+1}^{(\pm,q)}) = 0, \quad (23)$$

with

$$a_j^{(\pm)} = \omega_f j \pm \frac{\omega_0}{2} (-1)^j, \quad (24)$$

where the coefficient $c_k^{(\pm,q)}$ is the k th coefficient of the q th collective mode corresponding to the proper value $\omega(q)$. These coefficients are given by,

$$c_j^{(\pm,q)} = \prod_{k=0}^{j-1} s_k^{(\pm,q)} c_0^{(\pm,q)}, \quad (25)$$

where we have used the continued fraction

$$s_j^{(\pm,q)} = \frac{c_{j+1}^{(\pm,q)}}{c_j^{(\pm,q)}}, \quad (26)$$

$$= \frac{\lambda}{\omega(q) - a_{j+1}^{(\pm)} - \lambda s_{j+1}^{(\pm,q)}}, \quad (27)$$

$$= \frac{\lambda}{\omega(q) - a_{j+1}^{(\pm)} - \frac{\lambda^2}{\omega(q) - a_{j+2}^{(\pm)} - \frac{\lambda^2}{\omega(q) - a_{j+3}^{(\pm)} - \dots}}}, \quad (28)$$

where one can always set $c_0 = 1$ and normalize the semi-infinite set later. Notice that in the weak-coupling case a normal mode equivalent to that found in the rotating wave approximation treatment is recovered as the continued fraction is cut at the second term due to the parameter λ^2 being negligible compared to the frequency of the field.

In any given regime, we can take the continued fraction result and use it to write the eigenvectors of the quantum Hamiltonian in the reduced form,

$$|e_{\pm,m}\rangle = \sum_{j=0}^{\infty} c_j^{(\pm,m)} |\pm, j\rangle, \quad (29)$$

$$= \tilde{c}_0 \prod_{k=0}^{\hat{n}-1} s_k^{(m)} \hat{n}! e^{\hat{B}^\dagger - \hat{B}} |\pm, 0\rangle, \quad (30)$$

where \tilde{c}_0 is chosen such that $\langle e_{\pm,m} | e_{\pm,n} \rangle = \delta_{m,n}$.

V. PROPAGATION EXAMPLES

The dressed state basis that diagonalize our model, Eq.(11), implies that starting in a state of the kind $|j, e\rangle$ or $|j + 1, g\rangle$ with $j \geq 0$ will produce coherent oscillations. Such an initial state translates to laser light impinging the j th or $(j + 1)$ th waveguide of one of the lattices; the case j even (odd) corresponds to the crystal H_- (H_+). Figure 3(a) shows the propagation of light impinging at the 0th waveguide of our photonic crystal H_- in the weak-coupling regime which is equivalent to an initial state $|0, e\rangle$ in the quantum optics model. Accordingly, we observe the intensity oscillate between the 0th and first wave corresponding to an oscillation between the $|0, e\rangle$ and $|1, g\rangle$ states in the quantum optics model. Figure 3(c) presents a detail of the intensity at the 0th and first waveguide. This normalized intensity at the 0th waveguide is proportional to the probability of finding the time evolution of the quantum system back in the initial state, $I_0 \leftrightarrow P_{-,0}$, with

$$P_{-,0} = |\langle -, 0 | \psi(t) \rangle|^2, \quad |\psi(0)\rangle = |-, 0\rangle. \quad (31)$$

In quantum optics literature, it is known that the quantum Rabi model produces coherent oscillations in the two-level system inversion for $\omega_0 = 0$.²² In our model, due to the continuous spectra, all proper states are scattering states and at most we observe partial recovery of the original state when the field starts localized at a given waveguide. Figure 3(b) shows the propagation of light impinging at the 0th waveguide of the photonic lattice H_+ in the strong-coupling regime, which is equivalent to an initial state $|0, g\rangle$ in the quantum optics model; Figure 3(d) focus on the intensity at the first two waveguides. Again, the normalized intensity at the 0th waveguide is proportional to the probability of finding the time evolution of the quantum system back in the initial state, $I_0 \leftrightarrow P_{+,0} = |\langle +, 0 | \psi(t) \rangle|^2$, with $|\psi(0)\rangle = |+, 0\rangle$, for this case.

In both weak- and strong-coupling simulations a finite photonic lattice of size 5000 were used to produce the numerical results.

VI. CONCLUSION

We have proposed a set of two photonic crystals that classically simulates a new radiation-matter interaction where a two-level system is driven by the phase of a quantum field. We show that it is possible to determine exactly the dispersion relation of the photonic

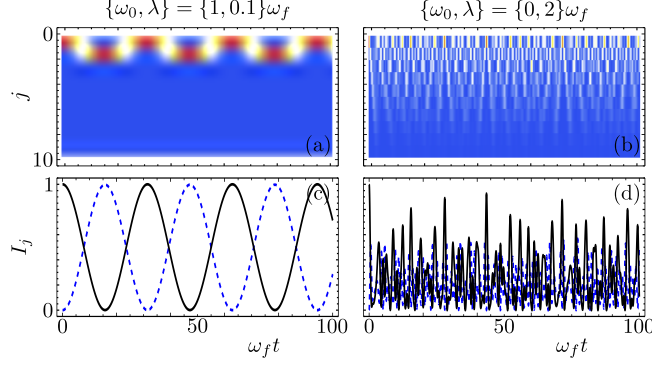


FIG. 3. (Color online) Examples of propagation in our positive parity photonic crystal. The left column (a,c) shows the case of weak-coupling, $\lambda = 0.1 \omega_f$, and the right (b,d) the case of strong-coupling, $\lambda = 2 \omega_f$. The first row (a,b) depicts intensity propagation on the first ten waveguides of a total of five thousand when light impinges the zeroth waveguide. The second row (c,d) shows the normalized intensity at the 0th (solid black) and first (dashed blue) waveguides. The dimensionless time parameter $\omega_f t$ is equivalent to the typical dimensionless propagation parameter.

waveguide lattices in the so-called weak-coupling regime and that in the strong-coupling regime we can use perturbation theory to approximate the dispersion relation up to second order perturbation. In the first case, the dispersion relation is discrete and, in the latter, continuous. The normal modes of the crystals are easily expressed in terms of continued fractions as a function of the dispersion relation. Our optical realization of a phase driven two-level system provides a scheme to explore an interesting process that is not accessible by usual means; i.e. cavity-QED or trapped ions.

* blas.rodriguez@gmail.com

¹ S. Longhi, Phys. Rev. Lett. **97**, 110402 (2006).

² H. B. Perets, Y. Lahini, F. Pozzi, M. Sorel, R. Morandotti, and Y. Silberberg, Phys. Rev. Lett. **100**, 170506 (2008).

³ Y. Bromberg, Y. Lahini, R. Morandotti, and Y. Silberberg, Phys. Rev. Lett. **102**, 253904 (2009).

⁴ F. Dreisow, A. Szameit, M. Heinrich, T. Pertsch, S. Nolte, A. Tunnermann, and S. Longhi, Phys. Rev. Lett. **102**, 076802 (2009).

- ⁵ Y. Lahini, Y. Bromberg, D. N. Christodoulides, and Y. Silberberg, *Phys. Rev. Lett.* **105**, 163905 (2010).
- ⁶ F. Dreisow, M. Heinrich, R. Keil, A. Tunnermann, S. Nolte, S. Longhi, and A. Szameit, *Phys. Rev. Lett.* **105**, 143902 (2010).
- ⁷ S. Longhi, *Phys. Rev. B* **81**, 075102 (2010).
- ⁸ S. Longhi, *Opt. Lett.* **36**, 3248 (2011).
- ⁹ R. Keil, A. Perez-Leija, F. Dreisow, M. Heinrich, H. Moya-Cessa, S. Nolte, D. N. Christodoulides, and A. Szameit, *Phys. Rev. Lett.* **107**, 103601 (2011).
- ¹⁰ B. M. Rodríguez-Lara, *Phys. Rev. A* **84**, 053845 (2011).
- ¹¹ S. Longhi, *Appl. Phys. B* **104**, 453 (2011).
- ¹² S. Longhi, *Opt. Lett.* **36**, 3407 (2011).
- ¹³ E. T. Jaynes and F. W. Cummings, *Proc. IEEE* **51**, 89 (1963).
- ¹⁴ H. Walther, B. T. H. Varcoe, B.-G. Englert, and T. Becker, *Rep. Prog. Phys.* **69**, 1325 (2006).
- ¹⁵ H. M. Moya-Cessa, Francisco Soto-Eguibar, *Physics Reports* **513**, 229 (2012).
- ¹⁶ A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, *Phys. Rev. A* **69**, 062320 (2004).
- ¹⁷ A. Kundu, *J. Phys. A: Math. Gen.* **37**, 281 (2004).
- ¹⁸ O. de los Santos-Sanchez and J. Récamier, *J. Phys. B: At. Mol. Opt. Phys.* **45**, 015502 (2012).
- ¹⁹ B. Buck and C. V. Sukumar, *Phys. Lett.* **81**, 132 (1981).
- ²⁰ E. A. Tur, *Opt. Spectrosc.* **89**, 574 (2000).
- ²¹ E. A. Tur, *Opt. Spectrosc.* **91**, 899 (2001).
- ²² J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, *Phys. Rev. Lett.* **105**, 263603 (2010).
- ²³ L. Susskind and J. Glogower, *Physics* **1**, 49 (1964).